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The Stokes resistance for a nearly spherical body

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Abstract. In order to calculate the Brownian motion of irregularly shaped objects, a first step is the solution for the Stokes flow past such objects, and a first step in this calculation is the development of a perturbation solution in the departure from spherical shape. A suitable variational approach is developed, from which it is shown that the expansion proceeds via a Green function which is obtained explicitly. In order to study statistical problems the expansion must be taken at least to second order in the departure from sphericity, and explicit solutions are given to this order. The analysis basically involves studies of the 3–*j* symbols of spherical harmonic analysis.

1. Introduction

As the first step in the study of Brownian motion or irregular objects two limiting cases come to mind. Firstly, that of a body which is nearly spherical, i.e. has a surface

$$r = a + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \epsilon_{lm} Y_{lm}(\theta, \phi) \tag{1.1}$$

where

$$\epsilon_{lm}^* = (-1)^m \epsilon_{l-m} \tag{1.2}$$

to ensure reality and $|\epsilon_{lm}| \ll a$. This problem is studied in this paper. The other, that of a cylinder whose axis is a random-flight trajectory, will be studied in a subsequent paper. The simplest attack on the flow past the surface (1.1) is to develop the fluid velocity as a series in the ϵ. Rather surprisingly this attitude seems only to have recently been considered. Erma (1963) gives a systematic discussion of the potential arising from a non-spherical charge distribution, but the Stokes problem is considerably complicated by the vector nature of the velocity field. The first systematic attack was made by Brenner (1964), using Lamb's general solution of Stokes' equation (Lamb 1932), and he gave an explicit formula to first order in the ϵ. The calculation to higher order, though in principle routine, becomes highly involved. The present paper develops the solution via a Routhian for the problem from which a matric Green function is developed as the iterative function, and by means of the use of 'angular momentum' analysis in terms of 3–*j* symbols the second-order solution is derived explicitly. Thus expressions are obtained which have a non-trivial average behaviour when, for example, the ϵ are determined by the thermal distribution of shapes of a body which is spherical at absolute zero. This is of course only a first step to a full treatment of the problem. For convenience in the calculations one may consider the body stationary (instead of translating in the liquid with velocity **V**) and the liquid flowing with a streaming velocity $-\mathbf{V}$. In this picture Stokes' equations are

$$\eta \nabla^2 \mathbf{v} - \nabla P = 0 \tag{1.3a}$$

$$\nabla . \mathbf{v} = 0. \tag{1.3b}$$

The last equation expresses the incompressibility condition of the fluid. The boundary conditions are as follows: on the particle surface

$$\lim_{\mathbf{r} \to \text{ to any point on } S} \mathbf{v}(\mathbf{r}) = 0 \tag{1.4a}$$

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and at infinity

$$\lim_{\mathbf{r}\to\infty} \mathbf{v}(\mathbf{r}) = -\mathbf{V} \tag{1.4b}$$

Having obtained the solution of Stokes' equations (1.3*a*) and (1.3*b*) appropriate to the boundary conditions (1.4*a*) and (1.4*b*) we find the Stokes resistance through the formula for the force on the surface S:

$$F_i = \int_S \sum_{\alpha=1}^{3} \sigma_{i\alpha}\, dS_\alpha \qquad (\alpha = 1, 2, 3) \tag{1.5}$$

where dS_α are the components of the outward normal surface element of S and

$$\sigma_{i\alpha} = P\delta_{i\alpha} + \eta\left(\frac{\partial v_i}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_i}\right) \tag{1.5a}$$

is the stress tensor for an incompressible fluid (Landau and Lifshitz 1960). Furthermore, the stress tensor $\sigma_{i\alpha}$ is divergenceless throughout a fluid obeying Stokes' equations, and therefore by Gauss's theorem

$$-\int_S \sum_{\alpha=1}^{3} \sigma_{i\alpha}\, dS_\alpha + \int_{S_R} \sum_{\alpha=1}^{3} \sigma_{i\alpha}\, dS_\alpha = 0 \tag{1.6}$$

where S_R is any closed surface containing the body surface S. For convenience in the calculations we shall take S_R a spherical surface of radius R. From (1.5) and (1.6) it follows that

$$F_i = \int_{S_R} \sum_{\alpha=1}^{3} \sigma_{i\alpha}\, dS_\alpha. \tag{1.7}$$

From (1.7) we obtain the Stokes resistance $\mathbf{F}$ as

$$\mathbf{F} = -\Phi\mathbf{V} \tag{1.8}$$

where Φ is the expression for the translation friction tensor relative to the body system.

In § 2 we obtain the general solution of the equations of motion as a Fourier integral involving the undetermined function $\boldsymbol{\xi}(\theta, \phi)$.

In § 3 we apply the general solution obtained in § 2 in the case of the sphere $r = a$ and find that the Stokes resistance is proportional to the Y_{00} coefficient in the expansion of $\boldsymbol{\xi}(\theta, \phi)$ in spherical harmonics.

In § 4 we develop a perturbation method for the calculation of the solutions of the Stokes equations (1.3*a*) and (1.3*b*) appropriate to the conditions (1.4*a*) and (1.4*b*). Further, we establish an integral equation for the density function ${}^{(\lambda)}\boldsymbol{\xi}(\theta, \phi)$ corresponding to the λth correction, whose kernel is independent of the correction order, and find the Green matrix of this equation exactly.

In § 5 we establish that the λth term of the series for the Stokes resistance is related to the ${}^{(\lambda)}\boldsymbol{\xi}^{00}$ component of ${}^{(\lambda)}\boldsymbol{\xi}(\theta, \phi)$ by a simple proportionality relation. Furthermore, ${}^{(\lambda)}\boldsymbol{\xi}^{00}$ can be obtained from the velocity terms up to order $\lambda - 1$. This result was also obtained by Brenner in the case $\lambda = 1$. Finally, we recover Brenner's result for the first-order correction to the drag and produce a formula for the second-order correction in terms of the Green matrix and the unperturbed solution. This result is applicable to any order of perturbation.

2. Variation method for the solution of Stokes' equations

To fix the boundary conditions of the problem, we derive the general solution in a suitable form, using Hamilton's variation principle. Taking as Routhian density

$$\mathscr{R}(\mathbf{r}) = -\tfrac{1}{2}\eta\{(\nabla v_1)^2 + (\nabla v_2)^2 + (\nabla v_3)^2\} \tag{2.1}$$

and employing the incompressibility condition (1.3*b*), we reproduce Stokes' equations as follows. To take into account the condition $\nabla.\mathbf{v} = 0$, we add to the Routhian density (2.1) the term $P(\mathbf{r})\nabla.\mathbf{v}(\mathbf{r})$, where $P(\mathbf{r})$ is the Lagrange multiplier for this constraint. It will turn out that $P(\mathbf{r})$ is the hydrodynamic pressure.

Now let us consider the functional

$$I\{\mathbf{v}(\mathbf{r})\} = \int\left\{-\tfrac{1}{2}\eta \sum_{k=1}^{3} (\nabla v_2)^2 + P(\mathbf{r})\nabla.\mathbf{v}\right\} d^3\mathbf{r} \tag{2.2}$$

where the integration extends over the whole space of the fluid. By Hamilton's principle the functional variation $\delta I\{\mathbf{v}(\mathbf{r})\}$ is taken zero for every variation of the velocity $\delta\mathbf{v}(\mathbf{r})$, subject to the condition $\delta\mathbf{v}(\mathbf{r}) = 0$ on the boundaries of the fluid, i.e.

$$\delta I = \int\left(-\eta \sum_{k,j=1}^{3} \frac{\partial v_k}{\partial x_j}\frac{\partial \delta v_k}{\partial x_j} + P\frac{\partial \delta v_k}{\partial x_j}\delta_{jk}\right) d^3\mathbf{r} \tag{2.3}$$

for $\delta\mathbf{v} = 0$ on the boundaries of integration and arbitrary elsewhere. Integrating (2.3) by parts we have

$$\delta I = \int \sum_{k,j=1}^{3}\left(-\eta\frac{\partial v_k}{\partial x_j} + P\delta_{jk}\right)\delta v_k\, dS_j + \int \sum_{k,j=1}^{3}\left(\eta\frac{\partial^2 v_k}{\partial x_j^2} - \frac{\partial P}{\partial x_j}\delta_{jk}\right)\delta v_k\, d^3\mathbf{r} = 0 \tag{2.4}$$

where the integration in the first integral on the right-hand side of (2.4) extends over the surface bounding the fluid. However, this integral is zero since $\delta\mathbf{v}$ on the fluid boundary is zero, and so equation (2.3) takes the form

$$\int(\eta\nabla^2\mathbf{v} - \nabla P).\delta\mathbf{v}\, d^3\mathbf{r} = 0. \tag{2.5}$$

Since $\delta\mathbf{v}$ is arbitrary everywhere apart from the boundaries, there follow Stokes' equations (1.3*a*). This shows that the Routhian density (2.1), together with the incompressibility condition, leads to the correct equations of motion.

Let us now derive, through Hamilton's principle, the equations of motion of the fluid in the presence of the body bounded by the surface S. Let $\mathbf{r} = \boldsymbol{\alpha}(\Omega)$, where $\Omega = (\theta, \phi)$, be the vector form of equation (1.1). The presence of the body in the infinite fluid imposes another constraint, namely that of the fluid velocity $\mathbf{v}(\mathbf{r})$ being zero at each point $\boldsymbol{\alpha}(\Omega)$ of the surface S. This condition is expressed by adding to the integrand of (2.2) the term

$$\int \boldsymbol{\xi}(\Omega').\mathbf{v}(\mathbf{r})\delta\{\mathbf{r} - \boldsymbol{\alpha}(\Omega')\}\, d\Omega' \tag{2.6}$$

where $d\Omega' = \sin\theta'\, d\theta' d\phi'$ and the integration over Ω' extends over the whole solid angle 4π. $\boldsymbol{\xi}(\Omega)$ is the Lagrange multiplier to account for the presence of the body.

The functional of $\mathbf{v}(\mathbf{r})$ to consider now is

$$I_1\{\mathbf{v}(\mathbf{r})\} = \int\left[-\tfrac{1}{2}\eta \sum_{k=1}^{3} (\nabla v_k)^2 + P\nabla.\mathbf{v} + \int \boldsymbol{\xi}(\Omega').\mathbf{v}(\mathbf{r})\delta\{\mathbf{r} - \boldsymbol{\alpha}(\Omega')\}\, d\Omega'\right] d^3\mathbf{r} \tag{2.7}$$

Applying Hamilton's principle to the functional (2.7) and performing similar manipulations as with (2.2), we find

$$\eta\nabla^2\mathbf{v} - \nabla P + \int \boldsymbol{\xi}(\Omega')\delta\{\mathbf{r} - \boldsymbol{\alpha}(\Omega')\}\, d\Omega' = 0. \tag{2.8}$$

Equation (2.8) is identical with equation (2.5) everywhere outside the surface S, and furthermore takes account of the surface S.

Let us now solve the system of the equations of motion (2.8) and (1.3*b*).

Taking the divergence of equation (2.8) and taking into account condition (1.3*b*), we find the Poisson-type equation

$$\nabla^2 P = \nabla . \int \boldsymbol{\xi}(\Omega')\delta\{\mathbf{r}-\boldsymbol{\alpha}(\Omega')\}\, d\Omega'. \tag{2.9}$$

Then we have from equation (2.9) the solution for the pressure:

$$P(\mathbf{r}) = P_0 - \nabla . \frac{1}{(2\pi)^3}\int \boldsymbol{\xi}(\Omega')\delta\{\mathbf{r}'-\boldsymbol{\alpha}(\Omega')\}.\frac{\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}(\Omega')\}]}{k^2}\, d^3\mathbf{k}\, d^3\mathbf{r}'\, d\Omega' \tag{2.10}$$

where the integration over $\mathbf{r}'$, $\mathbf{k}'$ covers the whole $\mathbf{r}'$ and $\mathbf{k}'$ spaces. Integrating in equation (2.10) over $\mathbf{r}'$ we obtain the expression for the pressure in Fourier form:

$$P(\mathbf{r}) = P_0 - \nabla . \frac{1}{(2\pi)^3}\int \boldsymbol{\xi}(\Omega')\frac{\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}(\Omega')\}]}{k^2}\, d^3\mathbf{k}\, d\Omega'. \tag{2.11}$$

Substituting the solution (2.11) for the pressure in equation (2.8) for the velocity, we obtain another Poisson-type equation, from which we obtain the solution for the velocity, which satisfies the incompressibility condition

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}_0 + \frac{1}{(2\pi)^3\eta}\left\{\int \boldsymbol{\xi}(\Omega').\frac{\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}(\Omega')\}]}{k^2}\, d^3\mathbf{k}\, d\Omega' + \nabla\nabla . \int \boldsymbol{\xi}(\Omega')\frac{\exp[i\mathbf{k}'.\{\mathbf{r}'-\boldsymbol{\alpha}(\Omega')\}]}{k'^2}\, d^3\mathbf{k}'\, d\Omega' . \frac{\exp\{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')\}}{k^2}\, d^3\mathbf{k}\, d^3\mathbf{r}'\right\}. \tag{2.12}$$

With some further manipulations we arrive at the Fourier form of the expression for the velocity:

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}_0 + \frac{1}{(2\pi)^3\eta}\int\left(\frac{\boldsymbol{\xi}(\Omega')}{k^2} + \nabla\nabla . \frac{\boldsymbol{\xi}(\Omega')}{k^4}\right)\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}(\Omega')\}]\, d^3\mathbf{k}\, d\Omega'. \tag{2.13}$$

The solution (2.13) for the velocity is convenient to adapt the boundary conditions at infinity and the no-slip condition on the surface S by a suitable choice of the constants P_0, $\mathbf{v}_0$ and density function $\boldsymbol{\xi}(\Omega)$.

3. Flow past a sphere

The vector form of the equation of a sphere of radius a, with respect to a Cartesian system with origin at the centre of the sphere, is

$$\mathbf{r} = a\begin{bmatrix}\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta\end{bmatrix} = \boldsymbol{\alpha}_0(\Omega). \tag{3.1}$$

The problem now is to specify P_0, $\mathbf{v}_0$ and $\boldsymbol{\xi}(\Omega)$ in equations (2.11) and (2.13), so that the boundary conditions (1.4*a*) and (1.4*b*) are satisfied, when $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0(\Omega)$. Let us perform in this case the integration over $\mathbf{k}$ in formulae (2.11) and (2.13). We have

$$P(\mathbf{r}) = P_0 - \frac{i}{(2\pi)^3}\int \sum_{j=1}^{3}\frac{k_j}{k}\xi_j(\Omega')\frac{\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}_0(\Omega')\}]}{k^2}\, d^3\mathbf{k}\, d\Omega' \tag{3.2}$$

$$v_i(\mathbf{r}) = v_{0i} + \frac{1}{(2\pi)^3\eta}\int \sum_{j=1}^{3}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\xi_j(\Omega')\frac{\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}_0(\Omega')\}]}{k^2}\, d^3\mathbf{k}\, d\Omega'. \tag{3.3}$$

The $\mathbf{k}$ integration can be performed by expressing the $\mathbf{k}$ quantities involved in (3.2) and (3.3) in spherical harmonics and making use of their orthonormality properties.

The following formulae involving the normalized complex spherical harmonics will be employed (Edmonds 1957):

$$\exp[i\mathbf{k}.\{\mathbf{r}-\boldsymbol{\alpha}_0(\Omega')\}] = 16\pi^2 \sum_{lm} \sum_{l'm'} (-1)^{l'} i^{l+l'} j_l(kr) j_{l'}(k\alpha) \times Y^*_{lm}(\Omega k)\, Y_{lm}(\Omega)\, Y_{l'm'}(\Omega k)\, Y^*_{l'm'}(\Omega') \tag{3.4}$$

$$Y_{LM} Y_{JK} = \sum_{l=|L-J|}^{L+J} \sum_{m=-l}^{l} (-1)^m \begin{pmatrix} J & K \\ L & M \\ l & -m \end{pmatrix} Y_{lm} = \sum_{l=|L-J|}^{L+J} (-1)^{M+K} \begin{pmatrix} J & K \\ L & M \\ l, & -M-K \end{pmatrix} Y_{l,\,M+K} \tag{3.5}$$

where the symbol in front of Y_{lm} in (3.5) denotes the Gaunt coefficient defined by

$$\begin{pmatrix} J & K \\ L & M \\ l & m \end{pmatrix} = \int Y_{JK} Y_{LM} Y_{lm}\, d\Omega. \tag{3.5a}$$

The Gaunt coefficient in terms of the 3-*j* symbols is expressed as (Rotenberg *et al.* 1959)

$$\begin{pmatrix} J & K \\ L & M \\ l & m \end{pmatrix} = \left\{\frac{(2J+1)(2L+1)(2l+1)}{4\pi}\right\}^{1/2} \begin{pmatrix} J & L & l \\ K & M & m \end{pmatrix} \begin{pmatrix} J & L & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.5b}$$

Using the formulae

$$k_1 = k \sin\theta_k \cos\phi_k, \qquad k_2 = k\sin\theta_k \sin\phi_k, \qquad k_3 = k\cos\theta_k \tag{3.6}$$

it is easy to express the vector $\mathbf{k}/k$ and the matrix $\delta_{ij} - k_i k_j/k^2$ appearing in the expressions (3.2) and (3.3) for the pressure and velocity in terms of spherical harmonics. We have

$$\frac{\mathbf{k}}{k} = \sum_{\nu=-1}^{1} \boldsymbol{\beta}_\nu Y_{1\nu}(\Omega_k) \tag{3.7}$$

where

$$\boldsymbol{\beta}_{-1} = \frac{1}{2}\left(\frac{8\pi}{3}\right)^{1/2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \qquad \boldsymbol{\beta}_0 = \left(\frac{4\pi}{3}\right)^{1/2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{\beta}_1 = \frac{1}{2}\left(\frac{8\pi}{3}\right)^{1/2} \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix} \tag{3.7a}$$

$$(\boldsymbol{\beta}_\nu = (-1)^\nu \boldsymbol{\beta}^*_{-\nu})$$

and

$$\delta_{ij} - \frac{k_i k_j}{k^2} = \tfrac{2}{3} I + \sum_{j=-2}^{2} I_j Y_{2j}(\Omega_k) \tag{3.8}$$

where I is the 3×3 unit matrix and

$$I_{-2} = \left(\frac{2\pi}{15}\right)^{1/2} \begin{bmatrix} -1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad I_{-1} = \left(\frac{2\pi}{15}\right)^{1/2} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & -i & 0 \end{bmatrix}$$

$$I_0 = \frac{2}{3}\left(\frac{\pi}{5}\right)^{1/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{3.8a}$$

$$I_1 = (-1)^1 I_{-1}^{x}, \qquad I_{+2} = (-1)^2 I_{-2}^{x}.$$

It will be seen later on that the vectors $\boldsymbol{\beta}_\nu$ and the matrices I_j play a very important role in the perturbation expansions of the pressure–velocity field and the drag.

With the aid of (3.5) and (3.7), the expression (3.2) for the pressure, after the integration over Ω_k, takes the form

$$P(\mathbf{r}) = P_0 - \frac{2}{\pi}\int \sum_{\nu=-1}^{1} \tilde{\boldsymbol{\beta}}_\nu . \boldsymbol{\xi}(\Omega') \sum_{lm} (-1)^m \begin{pmatrix} 1 & \nu \\ l+1 & m-\nu \\ l & -m \end{pmatrix} Y_{lm}(\Omega) Y^*_{l+1,m-\nu}(\Omega') d\Omega' \times \int_0^\infty j_l(kr) j_{l+1}(ka) k\, dk$$

$$- \frac{2}{\pi}\int \sum_{\nu=-1}^{1} \tilde{\boldsymbol{\beta}}_\nu . \boldsymbol{\xi}(\Omega') \sum_{lm} (-1)^m \begin{pmatrix} 1 & \nu \\ l-1 & m-\nu \\ l & -m \end{pmatrix} Y_{lm}(\Omega) Y^*_{l-1,m-\nu}(\Omega') d\Omega' \times \int_0^\infty j_l(kr) j_{l-1}(ka) k\, dk \qquad (3.9)$$

where by $\tilde{\boldsymbol{\beta}}_\nu$ we denote the row vector made by transposing the column vector $\boldsymbol{\beta}_\nu$.

We are interested in the exterior problem, i.e. in the region $r > a$. In this case we have (Bateman 1960)

$$\int_0^\infty j_l(kr) j_{l+1}(ka) k\, dk = 0, \qquad \int_0^\infty j_l(kr) j_{l-1}(ka) k\, dk = \frac{\pi a^{l-1}}{2r^{l+1}} \qquad (3.10)$$

and so formula (3.9) for the pressure simplifies to

$$P(\mathbf{r}) = P_0 + \sum_{lm} \sum_{\nu=-1}^{1} \int \frac{a^{l-1}}{r^{l+1}} \alpha_\nu{}^{lm} \tilde{\boldsymbol{\beta}}_\nu . \boldsymbol{\xi}(\Omega') Y_{lm}(\Omega) Y^*_{l-1,m-\nu}(\Omega')\, d\Omega' \qquad (3.11)$$

where we have introduced the notation

$$\alpha_\nu{}^{lm} = (-1)^m \begin{pmatrix} 1 & \nu \\ l-1 & m-\nu \\ l & -m \end{pmatrix}. \qquad (3.11a)$$

Using the formulae

$$\int_0^\infty j_l(kr) j_{l+2}(ka)\, dk = 0, \qquad \int_0^\infty j_l(kr) j_l(ka)\, dk = \frac{\pi a^l}{2(2l+1) r^{l+1}}$$

$$\int_0^\infty j_l(kr) j_{l-2}(ka)\, dk = \frac{\pi a^{l-2}}{4r^{l-1}}\left(1 - \frac{a^2}{r^2}\right), \qquad r > a \qquad (3.12)$$

(Bateman 1960), we obtain from (3.3), in a similar manner as with the pressure, the general expression for the velocity in the region $r > a$:

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}_0 + \frac{1}{\eta} \sum_{lm} \sum_{j=-2}^{2} \int \left(\tfrac{2}{3}\delta_{0j} I + \beta_j{}^{lm} I_j\right) \frac{a^l}{(2l+1) r^{l+1}} Y_{lm}(\Omega) Y^*_{lm-j}(\Omega') \boldsymbol{\xi}(\Omega')\, d\Omega'$$

$$+ \frac{1}{\eta} \sum_{lm} \sum_{j=-2}^{2} \int \gamma_j{}^{lm} I_j \frac{a^{l-2}}{2r^{l-1}} \left(\frac{a^2}{r^2} - 1\right) Y_{lm}(\Omega) Y^*_{l-2,m-j}(\Omega') \boldsymbol{\xi}(\Omega')\, d\Omega' \qquad (3.13)$$

where we have introduced the notation

$$\beta_j{}^{lm} = (-1)^m \begin{pmatrix} 2 & j \\ l & m-j \\ l & -m \end{pmatrix}, \qquad \gamma_j{}^{lm} = (-1)^m \begin{pmatrix} 2 & j \\ l-2 & m-j \\ l & -m \end{pmatrix}. \qquad (3.13a)$$

In the case of the interior problem the roles of a and r in the formulae (3.10) and (3.12) are interchanged. The general solution in the whole region $r > 0$ is obtained by superimposing the interior and exterior solutions. The constants P_0 and $\mathbf{v}_0$ come from the interior solution and are the only terms, which can be used for the exterior problem, without blowing up at infinity. The introduction of these constants in the formulae (3.2) and (3.3) makes it possible, for our purpose, to avoid the calculation of the solution for the interior problem.

Let us now perform the integration over Ω' in the expressions for the pressure and velocity, (3.11) and (3.13). Assuming

$$\boldsymbol{\xi}(\Omega') = \sum_{lm} \boldsymbol{\xi}^{lm} Y_{lm}(\Omega') \tag{3.14}$$

and bearing in mind the orthonormality relations of the spherical harmonics, we find the expressions for the pressure and velocity:

$$P(\mathbf{r}) = P_0 + \sum_{lm} \sum_{\nu=-1}^{1} \frac{a^{l-1}}{r^{l+1}} \alpha_\nu{}^{lm}(\tilde{\boldsymbol{\beta}}_\nu . \boldsymbol{\xi}^{l-1,m-\nu}) Y_{lm}(\Omega) \tag{3.15}$$

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}_0 + \frac{1}{\eta} \sum_{lm} \sum_{j=-2}^{2} (\tfrac{2}{3}\delta_{0j} I + \beta_j{}^{lm} I_j) \frac{a^l}{(2l+1) r^{l+1}} \boldsymbol{\xi}^{l,m-j} Y_{lm}(\Omega)$$
$$+ \frac{1}{\eta} \sum_{lm} \sum_{j=-2}^{2} \gamma_j{}^{lm} I_j \frac{a^{l-2}}{2r^{l-1}} \left(\frac{a^2}{r^2} - 1\right) \boldsymbol{\xi}^{l-2,m-j} Y_{lm}(\Omega). \tag{3.16}$$

It is obvious that terms in the expressions (3.15) and (3.16) containing $\boldsymbol{\xi}^{-1,m-j}$, $\boldsymbol{\xi}^{-2,m-j}$ are zero since the associated Gaunt coefficients are zero.

Let us now adapt (3.16) to the boundary conditions (1.4a) and (1.4b). For convenience we shall adopt the notation $P(r, \Omega)$ and $\mathbf{v}(r, \Omega)$ for $P(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$, respectively. The condition $\lim \mathbf{v}(r, \Omega) = -\mathbf{V}$ as $r \to \infty$ is satisfied by taking $\mathbf{v}_0 = -\mathbf{V}$ since, as $r \to \infty$, all terms apart from $\mathbf{v}_0$ on the right-hand side of (3.16) go to zero. The condition $\lim \mathbf{v}(r, \Omega) = 0$ as $r \to a+0$ gives the following equation for $\boldsymbol{\xi}^{lm}$:

$$\frac{1}{\eta a} \sum_{lm} \sum_{j=-2}^{2} (\tfrac{2}{3}\delta_{0j} I + \beta_j{}^{lm} I_j) \frac{1}{2l+1} \boldsymbol{\xi}^{l,m-j} Y_{lm}(\Omega) = (4\pi)^{1/2} \mathbf{V} Y_{00}(\Omega) \tag{3.17}$$

from which it follows that

$$\boldsymbol{\xi}^{00} = \tfrac{3}{2}(4\pi)^{1/2} \eta a \mathbf{V}, \qquad \boldsymbol{\xi}^{lm} = 0 \qquad \text{for } (l, m) \neq (0, 0). \tag{3.18}$$

The constant P_0 in the expression (3.15) for the pressure is fixed by external pressure acting on the fluid at infinity, which we may assume to be zero.

Introducing (3.18) into (3.15) and (3.16), we obtain the expressions for the pressure and velocity fields around the sphere $r = a$:

$$P^{(\mathrm{sph})}(r, \Omega) = \sum_{\nu=-1}^{1} \frac{3}{2} \frac{\eta a}{r^2} (\tilde{\boldsymbol{\beta}}_\nu . \mathbf{V}) Y_{1\nu}(\Omega) \tag{3.19}$$

$$\mathbf{v}^{(\mathrm{sph})}(r, \Omega) = \left\{ -1 + \frac{a}{r} + \frac{3}{4}\left(\frac{a^3}{r^3} - \frac{a}{r}\right) \sum_{j=-2}^{2} I_j Y_{2j}(\Omega) \right\} \mathbf{V}. \tag{3.20}$$

Let us now compute the Stokes resistance to the sphere $r = a$, using formula (1.7). Since formula (1.7) holds for every R, however large, it follows that the only contributing terms of the stress tensor to the Stokes resistance are those of order $1/r^2$, or what is the same, the contributing terms of the pressure are of order $1/r^2$ and of the velocity $1/r$ (see formula (1.5a) for the stress tensor).

Denoting by $\hat{P}$, $\hat{v}_i$ and $\hat{\sigma}_{i\alpha}$ the contributing parts of the pressure, velocity and stress tensor, we have in terms of Cartesian coordinates (x_1, x_2, x_3)

$$\hat{p}^{(\mathrm{sph})} = \frac{3\eta a}{2r^2} \sum_{j=-2}^{2} \frac{x_j V_j}{r} \tag{3.21}$$

$$\hat{v}_i^{(\mathrm{sph})} = \frac{3}{4}\frac{a}{r} V_i + \frac{3}{4}\frac{a}{r} \sum_{j=1}^{3} \frac{x_i x_j}{r^2} V_j \tag{3.22}$$

$$\hat{\sigma}_{i\alpha} = -\frac{9}{2}\frac{\eta a}{r^2} \sum_{j=1}^{3} \frac{x_i x_\alpha x_j}{r^3} V_j \tag{3.23}$$

where, for the transformation from polar into Cartesian coordinates, we have used formulae (3.7) and (3.8).

Employing formula (1.7) for the Stokes resistance with $\hat{\sigma}_{i\alpha}$ given by (3.23) and taking into account that

$$\int \frac{x_i x_\alpha}{r^2} \, d\Omega = \frac{4\pi}{3} \delta_{i\alpha} \tag{3.24}$$

we obtain the result

$$F_i^{(\mathrm{sph})} = -6\pi a \eta V_i. \tag{3.25}$$

From (3.25) and (3.18) we find that the coefficient $\boldsymbol{\xi}^{00}$ of the solution for the sphere is related to the Stokes force by

$$\mathbf{F}^{(\mathrm{sph})} = -\boldsymbol{\xi}(4\pi)^{1/2(\mathrm{sph})}\boldsymbol{\xi}^{00}. \tag{3.26}$$

As we shall see later on, this result enables one to develop a perturbation series for the Stokes resistance from a knowledge of the velocity field only. This will be the object of the subsequent sections.

4. Perturbation method

We are seeking the expressions for the pressure and velocity field, around the body confined by the surface given in (1.1), in the form of a power series in ϵ_{lm}, i.e.

$$P = P^{(0)} + P^{(1)} + P^{(2)} + \ldots \tag{4.1a}$$

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \ldots \tag{4.1b}$$

where $P^{(0)}$, $\mathbf{v}^{(0)}$ is the solution of the equations of motion, satisfying the boundary conditions at infinity exactly, the conditions on the body surface being satisfied to zero order in ϵ_{lm}. $P^{(\lambda)}$, $\mathbf{v}^{(\lambda)}$ ($\lambda = 1, 2, \ldots$) are the corrections to first, second, ... order in ϵ_{lm} for the pressure and velocity.

We take as zero-order approximation for P and $\mathbf{v}$ the solutions (3.19) and (3.20) for the sphere $r = a$, i.e.

$$P^{(0)} = P^{(\mathrm{sph})}, \qquad \mathbf{v}^{(0)} = \mathbf{v}^{(\mathrm{sph})}. \tag{4.2}$$

To find $P^{(\lambda)}$, $\mathbf{v}^{(\lambda)}$ we argue as follows. $\mathbf{v}(r, \Omega)$ must satisfy the following conditions:

(i) $\mathbf{v}(r, \Omega)$ together with $P(r, \Omega)$ solves the equations of motion (1.3*a*) and (1.3*b*).

(ii) $\mathbf{v}(r, \Omega)$ satisfies the conditions (1.4*b*) at infinity.

(iii) $\mathbf{v}(r, \Omega)$ becomes zero on S.

These conditions can be satisfied by choosing for $P^{(\lambda)}$, $\mathbf{v}^{(\lambda)}$ ($\lambda = 1, 2, \ldots$) the expressions for the pressure and velocity given in (3.15) and (3.16) without the constant terms, i.e. with $P_0 = 0$, $\mathbf{v}_0 = 0$ and suitably determined $\boldsymbol{\xi}(\Omega)$. We shall denote by ${}^{(\lambda)}\xi(\Omega)$ the $\boldsymbol{\xi}(\Omega)$ associated with the λth correction. Condition (i) is obviously satisfied since $P^{(\lambda)}$, $\mathbf{v}^{(\lambda)}$ solve the equations of motion, and so does any linear combination of them. Here we must note that, although $P^{(\lambda)}$, $\mathbf{v}^{(\lambda)}$ were derived on the assumption $r > a$, they hold as solutions for $r \leqslant a$, so that they can be used to describe the pressure–velocity field, slightly inside the surface $r = a$ as demanded by equation (1.1). Condition (ii) is satisfied by the zero-order term $\mathbf{v}^{(0)}$ and is not violated by adding any of the $v^{(\lambda)}$ since $v^{(\lambda)}(r, \Omega) \to 0$ as $r \to \infty$.

Condition (iii), which in explicit form reads

$$0 = \mathbf{v}^{(0)}(r, \Omega)+\mathbf{v}^{(1)}(r, \Omega)+\mathbf{v}^{(2)}(r, \Omega)+ \ldots \Big|_{r=a+\sum\limits_{LM}\epsilon_{LM}Y_{LM}(\Omega)} \tag{4.3}$$

gives rise to the equations for the determination of ${}^{(\lambda)}\boldsymbol{\xi}(\Omega)$ as follows.

Expanding (4.3) about $r = a$, we have

$$\begin{aligned} 0 = \mathbf{v}^{(0)}(a, \Omega)+\left(\frac{\partial \mathbf{v}^{(0)}}{\partial r}\right)_{r=a} \sum_{LM} \epsilon_{LM} Y_{LM}+\frac{1}{2}\left(\frac{\partial^2 \mathbf{v}^{(0)}}{\partial r^2}\right)_{r=a} \sum_{LM} \sum_{L'M'} \epsilon_{LM}\epsilon_{L'M'} Y_{LM} Y_{L'M'}+ \ldots \\ +\mathbf{v}^{(1)}(a, \Omega)+\left(\frac{\partial \mathbf{v}^{(1)}}{\partial r}\right)_{r=a} \sum_{LM} \epsilon_{LM} Y_{LM}+\frac{1}{2}\left(\frac{\partial^2 \mathbf{v}^{(1)}}{\partial r^2}\right)_{r=a} \sum_{LM} \sum_{L'M'} \epsilon_{LM}\epsilon_{L'M'} Y_{LM} Y_{L'M'}+ \ldots \\ +\mathbf{v}^{(2)}(a, \Omega)+\left(\frac{\partial \mathbf{v}^{(2)}}{\partial r}\right)_{r=a} \sum_{LM} \epsilon_{LM} Y_{LM}+\frac{1}{2}\left(\frac{\partial^2 \mathbf{v}^{(2)}}{\partial r^2}\right)_{r=a} \sum_{LM} \sum_{L'M'} \epsilon_{LM}\epsilon_{L'M'} Y_{LM} Y_{L'M'}+ \ldots . \end{aligned} \tag{4.4}$$

$\mathbf{v}^{(0)}(a, \Omega) = 0$ since $\mathbf{v}^{(0)}(r, \Omega)$ is the velocity for the sphere $r = a$. Bearing in mind that $\mathbf{v}^{(\lambda)}$ is of λth order in ϵ_{lm}, we have, arranging (4.4) in ascending powers of ϵ_{LM}, the equations

$$\mathbf{v}^{(1)}(a, \Omega) = \left(\frac{\partial \mathbf{v}^{(0)}}{\partial r}\right)_{r=a} \sum_{LM} \epsilon_{LM} Y_{LM} = {}^{(0)}\mathbf{W}(\Omega) \tag{4.5a}$$

$$\begin{aligned} \mathbf{v}^{(2)}(a, \Omega) = -\frac{1}{2}\left(\frac{\partial^2 \mathbf{v}^{(0)}}{\partial r^2}\right)_{r=a} \sum_{LM} \sum_{L'M'} \epsilon_{LM}\epsilon_{L'M'} Y_{LM} Y_{L'M'} \\ -\left(\frac{\partial \mathbf{v}^{(1)}}{\partial r}\right)_{r=a} \sum_{LM} \epsilon_{LM} Y_{LM} = {}^{(1)}\mathbf{W}(\Omega) \end{aligned} \tag{4.5b}$$

and so on.

From the equation (4.4*a*) we find ${}^{(1)}\boldsymbol{\xi}(\Omega)$, which can be used for the determination of $\mathbf{v}^{(1)}$. In a similar manner, having found $\mathbf{v}^{(1)}$, we can proceed, finding from equation (4.4*b*) ${}^{(2)}\boldsymbol{\xi}(\Omega)$, which leads to $\mathbf{v}^{(2)}$ and so on. The corresponding corrections for the pressure $P^{(1)}, P^{(2)} \ldots$, are also obtained from ${}^{(1)}\boldsymbol{\xi}(\Omega)$, ${}^{(2)}\boldsymbol{\xi}(\Omega)$, ... by utilizing formula (3.15) with $P_0 = 0$.

At this stage it is useful to introduce the solution

$$\sum_{lm} \sum_{j=-2}^{2} (\tfrac{2}{3}\delta_{0j} I+\beta_j{}^{lm} I_j)\frac{1}{2l+1} Y_{lm}(\Omega) Y^*_{l,m-j}(\Omega') = A(\Omega|\Omega'). \tag{4.6}$$

With the aid of (4.6) the λth equation (4.5) can be written

$$\int A(\Omega|\Omega')^{(\lambda)}\boldsymbol{\xi}(\Omega')\, d\Omega' = \eta a^{(\lambda-1)}\mathbf{W}(\Omega) \tag{4.7}$$

since

$$\lim \mathbf{v}^{(\lambda)}(r, \Omega) = \int \frac{1}{\eta a} A(\Omega|\Omega')\boldsymbol{\xi}(\Omega')\, d\Omega', \text{ as } r \to a$$

as one can easily see from (3.13).

The integral equation (4.7) plays a predominant role in the development of the perturbation series for the pressure–velocity field. The Green matrix $G(\Omega'|\Omega'')$ of equation (4.7), defined by the fundamental equation associated with it:

$$\int A(\Omega|\Omega') G(\Omega'|\Omega'')\, d\Omega' = I\delta(\Omega-\Omega'') = \sum_{LM} I Y_{LM}(\Omega) Y_{LM}(\Omega'') \tag{4.8}$$

determines ${}^{(\lambda)}\boldsymbol{\xi}(\Omega)$ through

$${}^{(\lambda)}\boldsymbol{\xi}(\Omega') = \eta a \int G(\Omega'|\Omega'')^{(\lambda-1)}\mathbf{W}(\Omega'')\, d\Omega''. \tag{4.9}$$

The function defined in (4.9) is easily seen to satisfy equation (4.7). By multiplying both members of equation (4.9) by $A(\Omega|\Omega')$ and integrating over Ω we obtain, by virtue of equation (4.8), on the right-hand side $\eta a^{(\lambda-1)}\mathbf{W}(\Omega)$.

To solve equation (4.8) we use as trial solution the matrix function

$$G(\Omega'|\Omega'') = \sum_{lm,m'} g^{lm,m'} Y_{lm}(\Omega') Y^*_{lm}(\Omega'') \tag{4.10}$$

where $g^{lm,m'}$ is a 3×3 matrix independent of Ω', Ω'', which we shall determine so that (4.10) satisfies equation (4.8).

Introducing (4.10) into (4.8) and performing the integration over Ω', we take

$$\sum_{LM,m} \sum_{j=-2}^{2} (\tfrac{2}{3}\delta_{0j} I + \beta_j{}^{LM} I_j) \frac{1}{2L+1} g^{Lm,M-j} Y_{LM}(\Omega) Y^*_{Lm}(\Omega'') = \sum_{LM,m} I\delta_{mM} Y_{LM}(\Omega) Y^*_{Lm}(\Omega'') \tag{4.11}$$

from which, owing to the orthogonality of the spherical harmonics, we obtain the displacement equations in M:

$$\sum_{j=-2}^{2} (\tfrac{2}{3}\delta_{0j} I + \beta_j{}^{LM} I_j) g^{Lm,M-j} = (2L+1)\delta_{mM} I \tag{4.12}$$

$$(L = 0, 1, 2, \ldots);\ (m, M = -L, (-L+1), \ldots, L-1, L).$$

Using the expressions (3.8a) for the matrices I_j and introducing the notation

$$\beta_{\pm 2}{}'^{LM} = \left(\frac{2\pi}{15}\right)^{1/2} \beta_{\pm 2}{}^{LM}, \qquad \beta_{\pm 1}{}'^{LM} = \left(\frac{2\pi}{15}\right)^{1/2} \beta_{\pm 1}{}^{LM}, \qquad \beta_0{}'^{LM} = \frac{2}{3}\left(\frac{\pi}{5}\right)^{1/2} \beta_0{}^{LM} \tag{4.13}$$

we write the matrix equations in (4.19) in component form as follows:

$$\beta_{-2}{}'^{LM}(g_{1j}{}^{Lm,M+2} + ig_{2j}{}^{Lm,M+2}) - \beta_{-1}{}'^{LM} g_{3j}{}^{Lm,M+1} + (\tfrac{2}{3} + \beta_0{}'^{LM}) g_{1j}{}^{Lm,M} + \beta_1{}'^{LM} g_{3j}{}^{Lm,M-j} - \beta_2{}'^{LM}(g_{1j}{}^{Lm,M-2} - ig_{2j}{}^{Lm,M-2}) = (2L+1)\delta_{mM}\delta_{1j} \tag{4.14a}$$

$$\beta_{-2}{}'^{LM}(-ig_{1j}{}^{Lm,M+2} + g_{2j}{}^{Lm,M+2}) - i\beta_{-1}{}'^{LM} g_{3j}{}^{Lm,M+1} + (\tfrac{2}{3} + \beta_0{}'^{LM}) g_{2j}{}^{Lm,M} - i\beta_1{}'^{LM} g_{3j}{}^{Lm,M-1} + \beta_2{}'^{LM}(ig_{1j}{}^{Lm,M-2} + g_{2j}{}^{Lm,M-2}) = (2L+1)\delta_{mM}\delta_{2j} \tag{4.14b}$$

$$-\beta_{-1}{}'^{LM}(g_{1j}{}^{Lm,M+1} + ig_{2j}{}^{Lm,M+1}) + (\tfrac{2}{3} - 2\beta_0{}'^{LM}) g_{3j}{}^{Lm,M} + \beta_1{}'^{LM}(g_{1j}{}^{Lm,M-1} - ig_{2j}{}^{Lm,M-1}) = (2L+1)\delta_{mM}\delta_{3j} \tag{4.14c}$$

($j = 1, 2, 3$), the indices L, m, M run as in (4.12).

From (4.14a) and (4.14b) one can obtain

$$(\tfrac{2}{3} + \beta_0{}'^{LM})(g_{1j}{}^{Lm,M} + ig_{2j}{}^{Lm,M}) + 2\beta_1{}'^{LM} g_{3j}{}^{Lm,M-1} - \beta_2{}'^{LM}(g_{1j}{}^{Lm,M-2} - ig_{2j}{}^{Lm,M-2}) = (2L+1)\delta_{mM}(\delta_{1j} + i\delta_{2j}) \tag{4.15a}$$

and

$$-\beta_{-2}{}'^{LM}(g_{1j}{}^{Lm,M+2} + ig_{2j}{}^{Lm,M+2}) - 2\beta_{-1}{}'^{LM} g_{3j}{}^{Lm,M+1} + (\tfrac{2}{3} + \beta_0{}'^{LM})(g_{1j}{}^{Lm,M} ig_{2j}{}^{Lm,M}) = (2L+1)\delta_{mM}(\delta_{1j} - i\delta_{2j}). \tag{4.15b}$$

Solving equation (4.14c) with respect to $g_{3j}{}^{Lm,M}$, we have

$$g_{3j}{}^{Lm,M} = \frac{1}{\tfrac{2}{3} - 2\beta_0{}'^{LM}} \{(2L+1)\delta_{mM}\delta_{3j} + \beta_{-1}{}'^{LM}(g_{1j}{}^{Lm,M+1} + ig_{2j}{}^{Lm,M+1}) - \beta_1{}'^{LM}(g_{1j}{}^{Lm,M-1} - ig_{2j}{}^{Lm,M-1})\}. \tag{4.15c}$$

Introducing the transformation

$$Z_j^{Lm,M} = g_{1j}^{Lm,M} + ig_{2j}^{Lm,M}, \qquad W_j^{Lm,M} = g_{1j}^{Lm,M} - ig_{2j}^{Lm,M} \tag{4.16}$$

and eliminating $g_{3j}^{Lm,M-1}$ and $g_{3j}^{Lm,M+1}$ from equations (4.15*a*) and (4.15*b*) through equation (4.15*c*), we reach the result

$$Z_j^{Lm,M} = A^{LM} W_j^{Lm,M-2} + B_j^{Lm,M} \tag{4.17a}$$

$$W_j^{Lm,M} = C^{LM} Z_j^{Lm,M+2} + D_j^{Lm,M} \tag{4.17b}$$

where

$$A^{LM} = \left(\tfrac{2}{3} + \beta_0{}'^{LM} + \frac{\beta_{-1}{}'^{LM-1}\beta_1{}'^{LM}}{\frac{1}{3} - \beta_0{}'^{LM-1}}\right)^{-1} \left(\beta_2{}'^{LM} + \frac{\beta_1{}'^{LM}\beta_1{}'^{LM-1}}{\frac{1}{3} - \beta_0{}'^{LM-1}}\right) \tag{4.18a}$$

$$B_j^{Lm,M} = \left(\tfrac{2}{3} + \beta_0{}'^{LM} + \frac{\beta_{-1}{}'^{LM-1}\beta_1{}'^{LM}}{\frac{1}{3} - \beta_0{}'^{LM-1}}\right)^{-1} (2L+1)\left\{\delta_{mM}(\delta_{1j} + i\delta_{2j}) - \frac{\beta_1{}'^{LM}\delta_{m,M-1}\delta_{3j}}{\frac{1}{3} - \beta_0{}'^{LM-1}}\right\} \tag{4.18b}$$

$$C^{LM} = \left(\tfrac{2}{3} + \beta_0{}'^{LM} + \frac{\beta_{-1}{}'^{LM}\beta_1{}'^{LM+1}}{\frac{1}{3} - \beta_0{}'^{LM}}\right)^{-1} \left(\beta_{-2}{}'^{LM} + \frac{\beta_{-1}{}'^{LM}\beta_{-1}{}'^{LM+1}}{\frac{1}{3} - \beta_0{}'^{LM}}\right) \tag{4.18c}$$

$$D_j^{Lm,M} = \left(\tfrac{2}{3} + \beta_0{}'^{LM} + \frac{\beta_{-1}{}'^{LM}\beta_1{}'^{LM+1}}{\frac{1}{3} - \beta_0{}'^{LM}}\right) (2L+1)\left\{\delta_{mM}(\delta_{1j} - i\delta_{2j}) + \frac{\beta_{-1}{}'^{LM}\delta_{m,M+1}\delta_{3j}}{\frac{1}{3} - \beta_0{}'^{LM}}\right\}. \tag{4.18d}$$

It is easy now to decouple equations (4.17*a*) and (4.17*b*), and the result is

$$Z_j^{Lm,M} = \frac{B_j^{Lm,M} + A^{LM} D_j^{Lm,M-2}}{1 - A^{LM} C^{LM-2}} \tag{4.19a}$$

$$W_j^{Lm,M} = \frac{D_j^{Lm,M} + C^{LM} B_j^{Lm,M+2}}{1 - C^{LM} A^{LM+2}}. \tag{4.19b}$$

With the aid of transformation (4.16) and, for example, (4.15*c*) we express the matrix elements $g_{ij}^{Lm,M}$ in terms of the quantities A, B, C and D, which are given in terms of the Gaunt coefficients. We have

$$\begin{aligned} g_{1j}^{Lm,M} &= \tfrac{1}{2}(Z_j^{Lm,M} + W_j^{Lm,M}) \\ &= \frac{1}{2}\left(\frac{B_j^{Lm,M} + A^{LM} D_j^{Lm,M-2}}{1 - A^{LM} C^{LM-2}} + \frac{D_j^{Lm,M} + C^{LM} B_j^{Lm,M+2}}{1 - C^{LM} A^{LM+2}}\right) \end{aligned} \tag{4.20a}$$

$$\begin{aligned} g_{2j}^{Lm,M} &= \frac{1}{2i}(Z_j^{Lm,M} - W_j^{Lm,M}) \\ &= \frac{1}{2i}\left(\frac{B_j^{Lm,M} + A^{LM} D_j^{Lm,M-2}}{1 - A^{LM} C^{LM-2}} - \frac{D_j^{Lm,M} + C^{LM} B_j^{Lm,M+2}}{1 - C^{LM} A^{LM+2}}\right) \end{aligned} \tag{4.20b}$$

$$\begin{aligned} g_{3j}^{Lm,M} = \tfrac{1}{2}(\tfrac{1}{3} - \beta_0{}'^{LM})^{-1}\Big\{&(2L+1)\delta_{mM}\delta_{3j} + \beta_{-1}{}'^{LM} \frac{B_j^{Lm,M+1} + A^{LM+1} D_j^{Lm,M-1}}{1 - A^{LM-1} C^{LM-1}} \\ &- \beta_1{}'^{LM} \frac{D_j^{Lm,M-1} + C^{LM-1} B_j^{LM,M+1}}{1 - A^{LM-1} C^{LM-3}}\Big\}. \end{aligned} \tag{4.20c}$$

The expressions (4.20*a*), (4.20*b*) and (4.20*c*) for the matrix elements $g_{ij}^{Lm,M}$ solve the problem of finding the Green matrix function defined in (4.8) completely, and consequently the problem of finding the velocity–pressure field to any desired order, according to the technique developed in this section.

It is easy to see from equation (4.12) that

$$g_{ij}^{00,0} = \tfrac{3}{2}\delta_{ij}. \tag{4.21}$$

Next we shall proceed to find the perturbation expressions for the Stokes resistance.

5. The Stokes resistance

Introducing the series (4.1*a*) and (4.1*b*) for the pressure and velocity field outside the surface given by equation (1) into the right-hand side of equation (1.7) for the Stokes resistance, we obtain, owing to the linearity of the stress tensor in the pressure and velocity components, the result

$$\mathbf{F}_i = \mathbf{F}_i^{(0)} + \mathbf{F}_i^{(1)} + \mathbf{F}_i^{(2)} + \dots \tag{5.1}$$

where $\mathbf{F}^{(0)}$ is the force on the sphere $r = a$ and $\mathbf{F}^{(\lambda)}$ ($\lambda = 1, 2, \dots$) is the λth correction due to the terms $P^{(\lambda)}$, $\mathbf{v}^{(\lambda)}$. Applying the same reasoning as with the calculation of the resistance to the sphere, we find that the only contributing terms of the pressure correction $P^{(\lambda)}$ are of order $1/r^2$, and of the velocity $\mathbf{v}^{(\lambda)}$ are those of order $1/r$. From (3.15) and (3.16) we have for the contributing parts of pressure and velocity corrections the expressions

$$\hat{P}^{(\lambda)} = \sum_{\nu, m=-1}^{1} \frac{\alpha_\nu^{1m}}{r^2} (\tilde{\boldsymbol{\beta}}_\nu . {}^{(\lambda)}\boldsymbol{\xi}^{0,m-\nu}) Y_{1m}(\Omega) = \sum_{\nu=-1}^{1} \frac{1}{(4\pi)^{1/2}} \frac{(\tilde{\boldsymbol{\beta}}_\nu . {}^{(\lambda)}\boldsymbol{\xi}^{00})}{r^2} Y_{1\nu}(\Omega) \tag{5.2}$$

where we have utilized the properties of the Gaunt coefficient α_ν^{lm} in (3.11*a*),

$$\begin{aligned} \hat{\mathbf{v}}^{(\lambda)} &= \frac{2}{3\eta} \frac{{}^{(\lambda)}\boldsymbol{\xi}^{00}}{r} Y_{00}(\Omega) - \frac{1}{\eta} \sum_{m,j=-2}^{2} \gamma_j^{2m} I_j \frac{{}^{(\lambda)}\boldsymbol{\xi}^{0,m-j}}{2r} Y_{2m}(\Omega) \\ &= \frac{2}{3\eta} \frac{{}^{(\lambda)}\boldsymbol{\xi}^{00}}{r} Y_{00}(\Omega) - \frac{1}{2\eta(4\pi)^{1/2}} \frac{{}^{(\lambda)}\boldsymbol{\xi}^{00}}{r} \sum_{m=-2}^{2} I_m Y_{2m}(\Omega) \end{aligned} \tag{5.3}$$

where in (5.3) we have utilized the properties of the symbols (3.13*a*).

It is easy to see that the functional dependence of $\hat{P}^{(\lambda)}$, $\hat{\mathbf{v}}^{(\lambda)}$ on ${}^{(\lambda)}\boldsymbol{\xi}^{00}$ is the same for every order of perturbation ($\lambda = 0, 1, 2, \dots$). Then, owing to the linearity of the stress tensor $\hat{\sigma}_{i\alpha}^{(\lambda)}$ in $\hat{P}^{(\lambda)}$, $\hat{\mathbf{v}}^{(\lambda)}$ and the linearity of $\hat{P}^{(\lambda)}$, $\hat{\mathbf{v}}^{(\lambda)}$ in ${}^{(\lambda)}\boldsymbol{\xi}^{00}$, it follows that

$$\mathbf{F}^{(\lambda)} = \mathbf{K}^{(\lambda)} . \boldsymbol{\xi}^{00} \tag{5.4}$$

where $\mathbf{K}$ is a 3×3 matrix independent of λ. Comparing (5.4) with (3.26), we find

$$K = -(4\pi)^{1/2} I \tag{5.6}$$

and therefore

$$\mathbf{F}^{(\lambda)} = -(4\pi)^{1/2} {}^{(\lambda)}\boldsymbol{\xi}^{00}. \tag{5.7}$$

Thus the problem of finding the λth perturbation term of the Stokes resistance is reduced to that of finding the λth $\boldsymbol{\xi}^{00}$.

Writing ${}^{(\lambda)}\mathbf{W}(\Omega)$ in the form

$${}^{(\lambda)}\mathbf{W}(\Omega) = \sum_{lm} {}^{(\lambda)}\mathbf{W}^{lm} Y_{lm}(\Omega) \tag{5.8}$$

and utilizing (4.10) for the Green matrix function, we find from formula (4.9) for ${}^{(\lambda)}\boldsymbol{\xi}(\Omega)$ the result for the coefficient ${}^{(\lambda)}\boldsymbol{\xi}^{00}$:

$${}^{(\lambda)}\boldsymbol{\xi}^{00} = \eta a g^{00,0} \, {}^{(\lambda-1)}\mathbf{W}^{00} = \tfrac{3}{2}\eta a \, {}^{(\lambda-1)}\mathbf{W}^{00} \tag{5.9}$$

where we have used the relation (4.21). Combining (5.9) and (5.7) we obtain

$$\mathbf{F}^{(\lambda)} = -\tfrac{3}{2}\eta a (4\pi)^{1/2} \, {}^{(\lambda-1)}\mathbf{W}^{00}. \tag{5.10}$$

From formulae (5.10) and (4.5*a*) and (4.5*b*) it follows that the λth correction to the Stokes resistance comes from a knowledge of $\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \dots, \mathbf{v}^{(\lambda-1)}$. This result was also obtained by Brenner for the first-order correction.

Let us now calculate the first-order correction to the drag. To do this, we need ${}^{(0)}\mathbf{W}^{00}$. We have from (4.5a)

$$
{}^{(0)}\mathbf{W}(\Omega) = \sum_{LM} \frac{\epsilon_{LM}}{a} \mathbf{V} Y_{LM}(\Omega) + \tfrac{3}{2} \sum_{LM} \sum_{l=|L-2|}^{L+2} \sum_{j=-2}^{2} \frac{\epsilon_{LM}}{a} (-1)^{M+j} \times \begin{pmatrix} 2 & j \\ L & M \\ l & -M-j \end{pmatrix} I_j \mathbf{V} Y_{l,M+j}(\Omega). \tag{5.11}
$$

From (5.11) we easily find that

$$
{}^{(0)}\mathbf{W}^{00} = \frac{\epsilon_{00}}{a} \mathbf{V} + \tfrac{3}{2} \sum_{M=-2}^{2} \frac{\epsilon_{2M}}{a(4\pi)^{1/2}} (-1)^M I_{-M} \mathbf{V}. \tag{5.12}
$$

Without loss of generality we take $\epsilon_{00} = 0$, as this can be incorporated into the radius a in equation (1).

Employing the reality condition (1.2) for the coefficients and the definition (3.8a) for the matrices I_j we find by means of (5.10) that $\mathbf{F}^{(1)}$ is given by

$$
\mathbf{F}^{(1)} = -\frac{9}{4}\eta \left(\epsilon_{20} I_0 + 2\mathrm{Re} \sum_{M=1}^{2} (-1)^M \epsilon_{2M} I_{-M} \right) \mathbf{V}. \tag{5.13}
$$

This is an equivalent form of Brenner's result (1964).

So the first-order correction to the drag involves only the coefficients of the second harmonic.

For the second-order correction to the force we need the quantity ${}^{(1)}\mathbf{W}^{00}$, which we find utilizing equation (4.5b) as follows. We have the term of ${}^{(1)}\mathbf{W}(\Omega)$, due to the zero-order velocity,

$$
\begin{aligned}
-\frac{1}{2}\left(\frac{\partial^2 \mathbf{v}^{(0)}}{\partial r}\right)_{r=a} & \sum_{LM} \sum_{L'M'} \epsilon_{LM}\epsilon_{L'M'} Y_{LM}(\Omega) Y_{L'M'}(\Omega) \\
&= -\frac{1}{a^2} \mathbf{V} \sum_{LM} \sum_{L'M'} \epsilon_{LM}\epsilon_{L'M'} Y_{LM}(\Omega) Y_{L'M'}(\Omega) \\
&\quad - \frac{15}{a^2} \sum_{LM} \sum_{L'M'} \sum_{j=-2}^{2} I_j \mathbf{V} Y_{2j}(\Omega) Y_{LM}(\Omega) Y_{L'M'}(\Omega).
\end{aligned} \tag{5.14}
$$

The coefficient of Y_{00} in the linear expansion in Y_{lm} of the first term on the right-hand side of (5.14) comes from $L' = L$ and $M' = -M$, and it is

$$
-\frac{1}{(4\pi)^{1/2}} \sum_{LM} \frac{\epsilon_{LM}\epsilon_{L,-M}}{a^2} \mathbf{V}. \tag{5.14a}
$$

The second term on the right-hand side of (5.14) gives as coefficient of Y_{00} the expression

$$
-\frac{15}{(4\pi)^{1/2}} \sum_{LM} \sum_{k,j=-2}^{2} \frac{\epsilon_{LM}\epsilon_{L+k,-M-j}}{a^2} \begin{pmatrix} 2 & j \\ L & M \\ L+k & -M-j \end{pmatrix} I_j \mathbf{V}. \tag{5.14b}
$$

Here we have taken into account that only those $Y_{L'M'}$, Y_{LM} with L' differing from L at most by 2 can produce a spherical harmonic of degree 2 which can be combined with Y_{2j} to give Y_{00}.

It remains to find the coefficient of Y_{00} of the second term on the right-hand side of (4.5b).

From (3.16) and (4.5a) we have

$$\mathbf{v}^{(1)}(r,\Omega) = \sum_{lm} {}^{(0)}\mathbf{W}^{lm}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\Omega) + \frac{1}{\eta}\sum_{lm}\sum_{j=-2}^{2} \gamma_j^{lm} I_j\left(\frac{a^l}{r^{l+2}} - \frac{a^{l-2}}{r^l}\right){}^{(1)}\boldsymbol{\xi}^{l-2,m-j} Y_{lm}(\Omega). \tag{5.15}$$

By differentiating (5.15) with respect to r at $r = a$, we form the second part of ${}^{(1)}\mathbf{W}(\Omega)$ as

$$-\left(\frac{\partial \mathbf{v}^{(1)}}{\partial r}\right)_{r=a} \sum_{LM} \epsilon_{LM} Y_{LM}(\Omega) = \sum_{lm}\sum_{LM} \frac{l+1}{a}{}^{(0)}\mathbf{W}^{lm}\epsilon_{LM} Y_{LM}(\Omega) Y_{lm}(\Omega) + \frac{1}{\eta a^2}\sum_{lm}\sum_{LM}\sum_{j=-2}^{2} \epsilon_{LM}\gamma_j^{lm} I_j {}^{(1)}\boldsymbol{\xi}^{l-2,m-j} Y_{lm}(\Omega) Y_{LM}(\Omega). \tag{5.16}$$

From (5.16) we find that the coefficient of Y_{00} in the linear expansion in Y_{lm} is

$$\sum_{LM} \frac{L+1}{a}\epsilon_{LM}\frac{(-1)^M}{(4\pi)^{1/2}}{}^{(0)}\mathbf{W}^{L,-M} + \frac{1}{\eta a^2}\sum_{LM}\sum_{j=-2}^{2} \epsilon_{LM}\frac{(-1)^M}{(4\pi)^{1/2}}\gamma_j^{L,-M}\,{}^{(1)}\boldsymbol{\xi}^{L-2,-M-j}. \tag{5.16a}$$

Employing formula (4.9) we find that

$${}^{(1)}\boldsymbol{\xi}^{LM} = \eta a \sum_{m=-L}^{L} g^{LM,m}\,{}^{(0)}\mathbf{W}^{Lm}. \tag{5.17}$$

With the aid of formula (5.17) we are able to express the unknown quantities $\boldsymbol{\xi}^{lm}$ in (5.16d) in terms of the matrices $g^{lm,m'}$ and the vectors ${}^{(0)}\mathbf{W}^{lm}$. ${}^{(1)}\mathbf{W}^{00}$ is the sum of the expressions (5.14a), (5.14b) and (5.16a):

$${}^{(1)}\mathbf{W}^{00} = -\frac{1}{(4\pi)^{1/2}}\sum_{LM}\frac{\epsilon_{LM}\epsilon_{L,-M}}{a^2}\mathbf{V} - \frac{15}{(4\pi)^{1/2}}\sum_{LM}\sum_{k,j=-2}^{2}\frac{\epsilon_{LM}\epsilon_{L+k,-M-j}}{a^2}(-1)^j\begin{pmatrix} 2 & j \\ L & M \\ L+k & -M-j\end{pmatrix} I_j\mathbf{V} + \sum_{LM}\frac{L+1}{(4\pi)^{1/2}}\frac{\epsilon_{LM}}{a}(-1)^M\,{}^{(0)}\mathbf{W}^{L,-M} + \sum_{LM}\sum_{j=-2}^{2}\sum_{m=-L}^{L}\frac{\epsilon_{LM}}{a}\frac{(-1)^M}{(4\pi)^{1/2}}\gamma_j^{L,-M} g^{L-2,-M-j,m}\,{}^{(0)}\mathbf{W}^{Lm} \tag{5.18}$$

where in (5.18) we have used (5.17).

Combining (5.10) and (5.18) we obtain the expression for the second-order correction to the drag:

$$\mathbf{F}^{(2)} = -\tfrac{3}{2}\eta a\sum_{LM}\left\{\frac{\epsilon_{LM}\epsilon_{L,-M}}{a^2} + 15\sum_{k,j=-2}^{2}\frac{\epsilon_{LM}\epsilon_{L+k,-M-j}}{a^2}(-1)^j\begin{pmatrix} 2 & j \\ L & M \\ L+k & -M-j\end{pmatrix} I_j\right\}\mathbf{V} + \tfrac{3}{2}\eta a\sum_{LM}\frac{\epsilon_{LM}}{a}(-1)^M(L+1)^{(0)}\mathbf{W}^{L,-M} + \tfrac{3}{2}\eta a\sum_{LM}\sum_{j=-2}^{2}\sum_{m=-|L-2|}^{L+2}(-1)^M\gamma_j^{L,-M} g^{L-2,-M-j,m}\,{}^{(0)}\mathbf{W}^{L-2,m} \tag{5.19}$$

From (5.19) it follows that there is no contribution from the last term to the second-order correction of the drag from the terms $\epsilon_{1m} Y_{1m}$, since $g^{-1m,m'}$ is zero. The contribution from the harmonics of degree 2 involves only the matrix $g^{00,0}$ whose expression is very simple (see (4.21)).

Here we note that all the second-order correction to the drag is expressed in terms of quantities related to the flow round the sphere $r = a$ and the matrices $g^{LM,m}$ and I_j. This result is true for any order of correction.

The calculation of the drag and therefore of the friction matrix by means of (1.8), in the case of a deformed spherical body, provides a more realistic approach to the Brownian motion of spherical-like particles. The authors hope to tackle this matter in a later publication.

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